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TENTAMEN RELATIVISTIC QUANTUM MECHANICS

Monday 21-04-2008, 09.00-12.00

On the first sheet write your name, address and student number. Write your name on all other sheets.

This examination consists of four problems, with in total 18 parts. The 18 parts carry equal weight in determining the final result of this examination.

$\hbar = c = 1$. The standard representation of the 4×4 Dirac gamma-matrices is given by:

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \qquad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \qquad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}.$$

PROBLEM 1

A spinor field transforms under Lorentz transformations as

$$\psi'(x') = S(\Lambda)\psi(x)\,, \tag{1.1}$$

where Λ is the Lorentz transformation matrix and $x^{\mu\prime} = \Lambda^\mu{}_\nu x^\nu$.

1.1 Show the Dirac equation is covariant under Lorentz transformations if

$$S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu\,. \tag{1.2}$$

1.2 Determine the transformation of $\bar{\psi}(x)$ under Lorentz transformations.

1.3 Show that $\bar{\psi}(x)\psi(x)$ is invariant under Lorentz transformations if

$$S^{-1} = \gamma^0 S^\dagger \gamma^0\,. \tag{1.3}$$

1.4 The interaction term of the photon field and the Dirac field in quantum electrodynamics,

$$\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x)\,,$$

must be invariant under Lorentz transformations. How should the photon field $A_\mu(x)$ *transform to achieve this invariance?*

PROBLEM 2

The Lagrangian density for a free Dirac field is

$$\mathcal{L} = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x)\,. \tag{2.1}$$

2.1 Determine the equation of motion of ψ and of $\bar{\psi}$.

Consider the infinitesimal transformations

$$\delta\psi = i\theta\gamma^5\psi\,. \tag{2.2}$$

2.2 Determine the variation $\delta\bar{\psi}$ under the transformation (2.2).

2.3 Obtain the variation of $\mathcal{L}$, and show that it vanishes if and only if $m = 0$.

2.4 Show that the current

$$j_5^\mu \equiv \bar{\psi}\gamma^\mu\gamma^5\psi$$

satisfies $\partial_\mu j_5^\mu = 0$ (for $m = 0$) if the equations of motion of ψ and $\bar{\psi}$ hold.

2.5 Show that

$$Q_5 = \int d^3x j_5^0,$$

is a conserved quantity (for $m = 0$) if the fields and their derivatives go to zero at infinity.

PROBLEM 3

Consider the theory of a scalar field $\phi(x)$, with the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \tfrac{1}{2}m^2\phi(x)\phi(x).$$

3.1 What is the canonical momentum $\pi(x)$ corresponding to the coordinate $\phi(x)$?

3.2 Given that classically $\{\phi(t,\vec{x}), \pi(t,\vec{y})\}_{\rm PB} = \delta^3(\vec{x}-\vec{y})$, what is the result of the equal-time commutation relation

$$[\phi(x), \pi(y)]_{x^0=y^0} \tag{3.2}$$

for the quantum operators ϕ and π?

3.3 The operator $\phi(x)$ can be written in the form

$$\phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \left(a(k)e^{-ikx} + a^\dagger(k)e^{ikx}\right)_{k^0=\omega_k}.$$

Show that the commutation relations

$$[a(k), a^\dagger(p)] = \delta^3(\vec{k} - \vec{p}), \quad [a(k), a(p)] = [a^\dagger(k), a^\dagger(p)] = 0,$$

imply the result of (3.2).

3.4 Show that ($x^0 \neq y^0$!)

$$[\phi(x), \phi(y)] = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(e^{-ik(x-y)} - e^{+ik(x-y)}\right)_{k^0=\omega_k}. \tag{3.4}$$

3.5 Using the result of (3.4) evaluate

$$[\phi(x), \partial_{y^0}\phi(y)],$$

and show that in the limit $x^0 \to y^0$ the result of (3.2) is obtained.

PROBLEM 4

Consider a scattering process between four photons: $\gamma_1 + \gamma_2 \to \gamma_3 + \gamma_4$, with four-momenta $k_1^\mu, k_2^\mu, k_3^\mu, k_4^\mu$. The spatial momenta satisfy

$$\vec{k}_1 + \vec{k}_2 = 0.$$

4.1 Show that $\vec{k}_3 + \vec{k}_4 = 0$.

4.2 Show that the energies of the four photons are equal.

4.3 Choose a coordinate system such that the spatial momentum of γ_1 is

$$\vec{k}_1 = (k, 0, 0).$$

We now perform a Lorentz boost in the x^1 direction. On the momenta this acts as on the coordinates:

$$k_i^{0\,\prime} = \gamma(k_i^0 - vk_i^1), \quad k_i^{1\,\prime} = \gamma(-vk_i^0 + k_i^1), \quad k_i^{2\,\prime} = k_i^2, \quad k_i^{3\,\prime} = k_i^3,$$

where the index $i = 1, \ldots, 4$ indicates the four photons and $\gamma = 1/\sqrt{1 - v^2}$. Calculate $\vec{k}_3{}' + \vec{k}_4{}'$ in this new coordinate system.

4.4 In quantumelectrodynamics, where photons interact with electrons and positrons, the scattering of photons is possible. Draw the lowest order (in the fine-structure constant α) Feynman diagram(s) that contribute to this process.